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A Reaction-Diffusion Model for Moderately Interacting Particles

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We consider a nonlinear reaction-diffusion model: n Brownian particles move independently in R^d and eventually die. The interaction, of binary type, affects only the death rate. The radius of interaction goes to zero as the number of particles increases and we characterize a wide range of speeds at which the radius goes to zero. Within this range we show a law of large numbers for the empirical distributions of the alive particles. The limit is independent of the choice of the speed and it is characterized as the solution of a nonlinear reaction-diffusion equation.

KEY WORDS: Reaction-diffusion; Brownian motion; moderate interaction; law of large numbers.

1. INTRODUCTION

In this paper we study the simplest case of a nonlinear reaction-diffusion model: particles move independently; they perform Brownian motions in $\mathbf{R}^d$; the only reaction is a binary one; it leads to the death (disappearance) of one of two particles which are close to each other; the rate at which this happens depends only on the distance of the two reacting particles. Formally, if $\mathbf{x}_i(t)$ denotes the position of the ith particle at time t and n is the number of particles present at the beginning, then the ith particle dies at a rate

$$r_n(t, \mathbf{x}_i) = 1/n \sum_{j \neq i} q_n(\mathbf{x}_j(t) - \mathbf{x}_i(t))\, \xi_{j,n}(t) \tag{1.1}$$

where $\xi_{j,n}(t)$ is the indicator of particle j being alive at time t and q_n is a given positive function on $\mathbf{R}^d$.

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We are interested, as n tends to infinity, in a law of large numbers (or "propagation of chaos") for the empirical measure

$$M_n(t) = 1/n \sum_j \delta(\mathbf{x}_j(t))\, \xi_{j,n}(t) \tag{1.2}$$

In particular, we ask which dependence on n implies such a law of large numbers: we fix a function q on $\mathbf{R}^d$ satisfying

$$q \geqslant 0, \qquad \int q(x)\, dx = c \tag{1.3}$$

If $\{a_n\}$ is some sequence tending to infinity, we define

$$q_n(x) = (a_n)^d\, q(a_n x) \tag{1.4}$$

From heuristic considerations, the limit of $M_n(t)$, if it exists, is expected to be of the form $u(t, x)\, dx$, where u satisfies the nonlinear reaction-diffusion equation

$$(\partial/\partial t) u = 1/2\, \Delta u - cu^2 \tag{1.5}$$

The problem can now be stated as follows: for which choice of $\{a_n\}$ does the limit equation (1.5) hold? Or: at which speed is the radius of interaction between particles (if we think of q as the indicator of a ball) allowed to shrink in order to obtain (1.5)?

The main result is the theorem in Section 2, which states that, under slight technical assumptions, the following condition on $\{a_n\}$, independently of q, suffices for a law of large numbers:

$$\begin{array}{lll} \text{if} & d=1 & \{a_n\} \text{ goes to infinity, without any restriction} \\ \text{if} & d=2 & \log(a_n) = o(n) \\ \text{if} & d \geqslant 3 & a_n = o(n^{1/(d-2)}) \end{array} \tag{1.6}$$

A recent paper of Sznitman[6] (see also Lang and Nguyen[2]) considers the following variant of our model (due originally to Smoluchowski[7]): death occurs instantaneously whenever a particle approaches another by a distance ρ_n. He shows that the correct choice for ρ_n, in order to get a nontrivial limit equation of the form (1.5), is

$$\begin{array}{ll} d=2 & \rho_n = \exp(-c_1 n) \\ d \geqslant 3 & \rho_n = c_2 n^{-1/(d-2)} \end{array} \tag{1.7}$$

with some constants c_1 and c_2.

So we still might expect for our model a reaction-diffusion equation of the same type if we make the "extreme" choice $a_n = \rho_n^{-1}$, with ρ_n given by (1.7); but it is clear that then the coefficient c of the quadratic term in (1.5) will have to be replaced by a constant which reflects in a much finer way the geometry of q (in refs. 2 and 6 the Newtonian capacity of a ball was the decisive quantity). It is obvious from refs. 2 and 6 that if $\{\rho_n\}$ shrinks faster than (1.7), in the limit no reaction at all would take place, since the particles would no longer meet each other.

From these considerations it is evident that our model with $\{a_n\}$ given by (1.6) is a certain simplification of ref. 6, a step in the "mean-field" direction if we compare it with the variant $a_n = \rho_n^{-1}$, ρ_n given by (1.7), which would be more closely related to ref. 6. On the other hand, we think that our method of studying the accumulated risk process (see next paragraph) and showing its deterministic character in the limit is a natural tool to apply here; and since in the case $a_n = \rho_n^{-1}$ the risk process obviously would no longer converge to a deterministic one, we may say that in a certain sense our results are sharp. (Note also that the notion of risk process is meaningless for the model of Sznitman; death is defined there only through the trajectories, without an additional random mechanism.)

The reason why condition (1.6) plays a role may be understood by the following argument. One has to look at the "accumulated risk" of a particle

$$R_n(t, \mathbf{x}_i) = 1/n \sum_{j \neq i} \int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \xi_{j,n}(s)\, ds \tag{1.8}$$

and consider whether a law of large numbers for $R_n(t, \mathbf{x}_i)$ holds in the sense that it becomes, in the limit of large n, a deterministic function of the trajectory $\mathbf{x}_i(s)$, $s \leqslant t$, namely [see (4.1)]

$$R(t, \mathbf{x}_i) = c \int_0^t u(s, \mathbf{x}_i(s))\, ds \tag{1.9}$$

This calculation in a simplified form where all the $\xi_{j,n}(t)$ appearing in the definition (1.1) are replaced by 1 is carried out in Lemma 3.3 (take $v \equiv 0$); such a law holds if, for any fixed t,

$$\int_0^t ds \int q_n(y)\, dy \int q_n(z)\, dz\, p[2s, z-y] = o(n) \tag{1.10}$$

where $p[2s, z-y]$ is the transition density of the difference of two independent Brownian motions.

If, in contrast, one chooses a strategy of proof which involves the "hazard rate" $r_n(t, \mathbf{x}_i)$ (not the accumulated risk) and shows that it is close to a deterministic function of $\mathbf{x}_i$ for each t, one has to impose sharper conditions on $\{a_n\}$; see, e.g., ref. 1, where $d=1$, $a_n = \text{const} \cdot n$.

A last remark on the term "moderate interaction": we found that terminology in a paper of Oeschläger [5] (see also Méléard and Roelly-Coppoletta[(3)]), though in a somehow different context of an interaction causing an additional drift term for each Brownian motion instead of death. Also there, a wide range of speeds at which the radius of interaction goes to zero was characterized; it was shown that inside this range the limiting equation is always the same. It is easy to see that the limiting equation in refs. 3 and 5 as well as in our case would also show up if one first takes the Vlasov–McKean limit, in which the radius of interaction is fixed (corresponding, in our case, to $a_n \equiv 1$; compare Nappo and Orlandi[(4)]) and only then takes the limit in which the radius of interaction goes to zero (this corresponds, in our case, to q converging to a delta function through the sequence q_n, without any further restriction on $\{a_n\}$).

The paper is organized as follows: in Section 2 notations and assumptions are introduced and the theorem is stated. Auxiliary results needed for the proof of a more general character (for example, the basic Lemma 3.3) are contained in Section 3. In Section 4 we give the main steps of the proof of the theorem, whereas technical estimates are collected in Section 5.

2. DESCRIPTION OF THE MODEL AND RESULTS

The microscopic model we consider consists of a system of n random particles $(\mathbf{x}_i)_{i=1,\dots,n}$ which move as independent Brownian motions in $\mathbf{R}^d$ and die according to a rate which is of local mean-field type depending on the configuration of the system. Each particle is represented by the process $(\mathbf{x}_i, \xi_{i,n})_{i=1,\dots,n}$, where the $\xi_{i,n}$ are point processes taking values in $\{0, 1\}$ defining the state of the particles: dead or alive.

To define more precisely these processes, we introduce the probability space

$$\Omega = (\mathbf{C}([0, T], \mathbf{R}^d) \times \mathbf{R}^+)^\infty$$

with the canonical filtration and the probability

$$\mathbf{P}(\mathbf{d}\omega) = \bigotimes_{i=1,\infty} [P(d\omega_i) \otimes \exp(-\sigma_i)\, d\sigma_i] \tag{2.1}$$

with P the probability measure associated with the Brownian process with initial distribution $\pi_0(x)\, dx$. We denote by ω_i and σ_i the canonical variables: the ω_i are the independent Brownian processes and the σ_i are exponential times independent of each other and of the ω_i.

In this setting we define in $\mathbf{D}([0, T], \{0, 1\})$ the process $\xi_{i,n}$ as the strong solution of

$$\xi_{i,n}(t) = \mathbf{I}_{\{1/n \sum_{j \neq i} \int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s)) \cdot \xi_{j,n}(s)\, ds < \sigma_i\}} \tag{2.2}$$

where $\mathbf{I}_A$ is the indicator function of the set A and

$$q_n(x) = (a_n)^d\, q(a_n x) \tag{2.3}$$

with $\{a_n\}$ a divergent sequence and $q \in L^1 \cap H^{-1}$ a nonnegative function; here H^{-1} is the Sobolev space of functions g on $\mathbf{R}^d$ such that, denoting by $\hat{g}(\omega)$ the Fourier transform of g, $\int |\hat{g}(\omega)|^2 (1 + \omega^2)^{-1}\, d\omega$ is finite.

Remark. Note that, when $d = 1$, if q is integrable, then $q \in H^{-1}$ is automatically satisfied.

We set

$$c = \int q(x)\, dx \tag{2.4}$$

We define the processes ξ_i as

$$\xi_i(t) = \mathbf{I}_{\{R(t, \mathbf{x}_i) < \sigma_i\}} \tag{2.5}$$

where

$$R(t, \mathbf{x}_i) = c \int_0^t u(s, \mathbf{x}_i(s))\, ds \tag{2.6}$$

with u the solution of

$$(\partial/\partial t) u = 1/2\, \Delta u - c u^2; \qquad u(0, x) = \pi_0(x) \tag{2.7}$$

We will prove the following result for this model.

Theorem. Suppose that $\pi_0 \in L^\infty(\mathbf{R}^d)$, $q \geqslant 0$, $q \in L^1 \cap H^{-1}$, q_n defined as in (2.3), and $\{a_n\}$ a divergent sequence; moreover, if $d > 1$, assume that

$$\begin{aligned} &\text{if} \quad d = 2, \qquad \log(a_n) = o(n) \\ &\text{if} \quad d \geqslant 3, \qquad a_n = o(n^{1/(d-2)}) \end{aligned} \tag{2.8}$$

Then, for every $m \geqslant 1$,

$$(\mathbf{x}_i, \xi_{i,n})_{i=1,\dots,m} \to (\mathbf{x}_i, \xi_i)_{i=1,\dots,m} \qquad \text{as} \quad n \to \infty$$

in probability in the space $(\Omega, \mathbf{P})$, with respect to the Skorohod metric in $\mathbf{D}([0, T], \{0, 1\})$, for any fixed $T>0$.

Remark. Convergence in probability of $\xi_{i,n}$ to ξ_i is obviously equivalent to

$$\int_0^T \mathbf{E}[|\xi_{i,n}(t)-\xi_i(t)|]\,dt \to 0 \qquad \text{as} \quad n\to\infty \tag{2.9}$$

which implies that, for any $m\geq 1$ and $F_i \in \mathbf{C}_b(\mathbf{D}([0, T], \{0, 1\}))$

$$\mathbf{E}\Big[\bigotimes_{i=1,m} F_i(\xi_{i,n})\Big] \to \mathbf{E}\Big[\bigotimes_{i=1,m} F_i(\xi_i)\Big] \qquad \text{as} \quad n\to\infty$$

i.e., what is usually called propagation of chaos, since the $\mathbf{x}_i$ have been taken independent.

3. AUXILIARY RESULTS

For analyzing the problem it is convenient to set

$$R_n(t, \mathbf{x}_i) = 1/n \sum_{j\neq i} \int_0^t q_n(\mathbf{x}_j(s)-\mathbf{x}_i(s))\,\xi_{j,n}(s)\,ds \tag{3.1}$$

$R_n(t, \mathbf{x}_i)$ is a functional of the processes which counts the "integrated risk" felt by the particle with trajectory $\mathbf{x}_i$ up to time t. In this setting the definition (2.2) of the process $\xi_{i,n}$ becomes

$$\xi_{i,n}(t) = \mathbf{I}_{\{R_n(t, \mathbf{x}_i) < \sigma_i\}} \tag{3.2}$$

It is natural to achieve the convergence of $\xi_{i,n}$ through the convergence of the risk $R_n(t, \mathbf{x}_i)$ to the limit risk $R(t, \mathbf{x}_i)$, (2.6), which defines the limiting indicator of a particle being alive,

$$\xi_i(t) = \mathbf{I}_{\{R(t, \mathbf{x}_i) < \sigma_i\}}$$

The following lemma gives the relation between processes defined through risks functionals, and the risk functionals.

Lemma 3.1. Let σ be an exponential time and for $j=1, 2$ let S_j be nonnegative, nondecreasing, right continuous random functions. Define for $j=1, 2$ and $t\in[0, T]$

$$\eta_j(t) = \mathbf{I}_{\{S_j(t) < \sigma\}}$$

Then, for any $\varepsilon > 0$

$$|\eta_1(t) - \eta_2(t)| \leqslant 1/\varepsilon\, |S_1(t) - S_2(t)| + \mathbf{I}_{\{|S_1(t) - \sigma| \leqslant \varepsilon\}}$$

Moreover, if S_1 is independent of σ, then, for any $\varepsilon > 0$,

$$\mathbf{E}[|\eta_1(t) - \eta_2(t)|] \leqslant 1/\varepsilon\, \mathbf{E}[|S_1(t) - S_2(t)|] + 2\varepsilon$$

Remark. If S_1 and S_2 are both independent of σ, one has obviously the simpler relation

$$\mathbf{E}[|\eta_1(t) - \eta_2(t)|] \leqslant \mathbf{E}[|S_1(t) - S_2(t)|]$$

but in the sequel we need a relation between η_j and S_j without such an assumption.

Proof of Lemma 3.1. We have

$$\begin{aligned}
|\eta_1(t) - \eta_2(t)| &= \mathbf{I}_{\{S_1(t) < \sigma \leqslant S_2(t)\}} + \mathbf{I}_{\{S_2(t) < \sigma \leqslant S_1(t)\}} \\
&= \mathbf{I}_{\{|S_1(t) - S_2(t)| \leqslant \varepsilon\}} (\mathbf{I}_{\{S_1(t) < \sigma \leqslant S_2(t)\}} + \mathbf{I}_{\{S_2(t) < \sigma \leqslant S_1(t)\}}) \\
&\quad + \mathbf{I}_{\{|S_1(t) - S_2(t)| > \varepsilon\}} (\mathbf{I}_{\{S_1(t) < \sigma \leqslant S_2(t)\}} + \mathbf{I}_{\{S_2(t) < \sigma \leqslant S_1(t)\}}) \\
&\leqslant \mathbf{I}_{\{|S_1(t) - S_2(t)| \leqslant \varepsilon\}} (\mathbf{I}_{\{S_1(t) < \sigma \leqslant S_2(t)\}} + \mathbf{I}_{\{S_2(t) < \sigma \leqslant S_1(t)\}}) \\
&\quad + \mathbf{I}_{\{|S_1(t) - S_2(t)| > \varepsilon\}} \\
&\leqslant \mathbf{I}_{\{|S_1(t) - \sigma| \leqslant \varepsilon\}} + |S_1(t) - S_2(t)|/\varepsilon
\end{aligned}$$

Therefore we have that

$$\mathbf{E}[|\eta_1(t) - \eta_2(t)|] \leqslant 1/\varepsilon\, \mathbf{E}[|S_1(t) - S_2(t)|] + \mathbf{P}[|S_1(t) - \sigma| \leqslant \varepsilon]$$

Moreover, if F is the distribution function of $S_1(t)$, taking into account the independence of $S_1(t)$ and σ, we have

$$\mathbf{P}[|S_1(t) - \sigma| \leqslant \varepsilon] = \int_0^\infty dF(x) \int_{(x-\varepsilon) \vee 0}^{(x+\varepsilon)} e^{-s}\, ds \leqslant 2\varepsilon$$

In the following we will use the following generalization of Lemma 3.1.

Lemma 3.2. If S_j and η_j are defined as in Lemma 3.1 and if S satisfies the same hypotheses as S_j, then

$$|\eta_1(t) - \eta_2(t)| \leqslant 1/\varepsilon\, |S_1(t) - S(t)| + 1/\varepsilon\, |S_2(t) - S(t)| + \mathbf{I}_{\{|S(t) - \sigma| \leqslant \varepsilon\}}$$

Proof. The proof is similar to the previous one, considering that

$$|\eta_1(t) - \eta_2(t)| = (\mathbf{I}_{\{S_1(t) < \sigma \leqslant S_2(t)\}} + \mathbf{I}_{\{S_2(t) < \sigma \leqslant S_1(t)\}}) \\ \times (\mathbf{I}_{\{|S_1(t) - S(t)| \leqslant \varepsilon\}} + \mathbf{I}_{\{|S_1(t) - S(t)| > \varepsilon\}}) \\ \times (\mathbf{I}_{\{|S_2(t) - S(t)| \leqslant \varepsilon\}} + \mathbf{I}_{\{|S_2(t) - S(t)| > \varepsilon\}})$$

To understand the constraints (2.8) on $\{a_n\}$, we consider a simplified model: we consider a particle $\mathbf{x}_i$ in an environment where all the other particles $\mathbf{x}_j$ die independently of each other with a rate $v(t, \mathbf{x}_j(t))$ at time t. It is easy to see that, under these constraints, the risk $S_n(t, \mathbf{x}_i)$ felt by the particle $\mathbf{x}_i$ converges in $L^2(\Omega, \mathbf{P})$ to a limit risk which depends only on the trajectory $\mathbf{x}_i$ and does not depend on all the other trajectories, namely:

Lemma 3.3. Define in $(\Omega, \mathbf{P})$, for $j \neq i$,

$$\eta_j(t) = \mathbf{I}_{\{\int_0^t v(s, \mathbf{x}_j(s))\, ds < \sigma_j\}} \tag{3.3}$$

with $v \in L^\infty$ and nonnegative, and define

$$S_n(t, \mathbf{x}_i) = 1/n \sum_{j \neq i} \int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \eta_j(s)\, ds$$

Then, it $\{a_n\}$ satisfies the growth conditions (2.8),

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}\left[\left\{S_n(t, \mathbf{x}_i) - c \int_0^t u(s, \mathbf{x}_i(s))\, ds\right\}^2\right] \to 0 \qquad \text{as} \quad n \to \infty$$

where u is the solution of

$$(\partial/\partial t) u = 1/2\, \Delta u - vu; \qquad u(0) = \pi_0 \tag{3.4}$$

Proof. We have

$$\begin{aligned}
&\mathbf{E}\left[\left\{S_n(t, \mathbf{x}_i) - c \int_0^t u(s, \mathbf{x}_i(s))\, ds\right\}^2\right] \\
&\quad = 1/n^2 \sum_{j \neq i} \mathbf{E}\left[\left\{\int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \eta_j(s)\, ds\right\}^2\right] \\
&\qquad + 1/n^2 \sum_{h \neq i} \mathbf{E}\left[\int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \eta_j(s)\, ds \right. \\
&\quad \left. \times \int_0^t q_n(\mathbf{x}_h(r) - \mathbf{x}_i(r))\, \eta_h(r)\, dr\right] \\
&\qquad + \mathbf{E}\left[\left\{c \int_0^t u(s, \mathbf{x}_i(s))\, ds\right\}^2\right] \\
&\qquad - 2(1/n) \sum_{j \neq i} \mathbf{E}\left[\int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \eta_j(s)\, ds\; c \int_0^t u(r, \mathbf{x}_i(r))\, dr\right]
\end{aligned}$$

$$\leqslant 1/n\, \mathbf{E}\left[\left\{\int_0^t q_n(\mathbf{x}_j(s)-\mathbf{x}_i(s))\,\eta_j(s)\,ds\right\}^2\right]$$

$$+(n-1)/n\, \mathbf{E}\left[\mathbf{E}\left[\int_0^t q_n(\mathbf{x}_j(s)-\mathbf{x}_i(s))\,\eta_j(s)\,ds\,\middle|\,\mathbf{x}_i\right]\right.$$

$$\left.\times\mathbf{E}\left[\int_0^t q_n(\mathbf{x}_h(r)-\mathbf{x}_i(r))\,\eta_h(r)\,dr\,\middle|\,\mathbf{x}_i\right]-\left\{c\int_0^t u(s,\mathbf{x}_i(s))\,ds\right\}^2\right]$$

$$+2(n-1)/n\, \mathbf{E}\left[c\int_0^t u(s,\mathbf{x}_i(s))\,ds\left\{c\int_0^t u(s,\mathbf{x}_i(s))\,ds\right.\right.$$

$$\left.\left.-\mathbf{E}\left[\int_0^t q_n(\mathbf{x}_j(s)-\mathbf{x}_i(s))\,\eta_j(s)\,ds\,\middle|\,\mathbf{x}_i\right]\right\}\right]$$

$$+1/n\, \mathbf{E}\left[\left\{c\int_0^t u(s,\mathbf{x}_j(s))\,ds\right\}^2\right]$$

Let us denote by $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, $\mathbf{I}_4$ the four terms in this expression. The quantity $\mathbf{I}_1$ can be expressed in terms of $\{a_n\}$:

$$\mathbf{E}\left[\left\{\int_0^t q_n(\mathbf{x}_1(s)-\mathbf{x}_2(s))\,ds\right\}^2\right] \begin{array}{ll} \text{is bounded} & \text{for } d=1 \\ =O(\log a_n) & \text{for } d=2 \\ =O(a_n^{d-2}) & \text{for } d\geqslant 3 \end{array}$$

Indeed, letting $p[s, y-x]$ be the transition probability of a standard Brownian motion, and noting that $\mathbf{x}_1-\mathbf{x}_2$ is a Brownian motion with diffusion coefficient 2 and initial probability density $\pi_*=\pi_0*\bar{\pi}_0$ ($\bar{\pi}_0$: π_0 reflected at 0), which is still bounded by $\|\pi_0\|_\infty$, we have

$$\begin{aligned}
&\mathbf{E}\left[\left\{\int_0^t q_n(\mathbf{x}_1(s)-\mathbf{x}_2(s))\,ds\right\}^2\right]\\
&\quad=2\mathbf{E}\left[\int_0^t q_n(\mathbf{x}_1(s)-\mathbf{x}_2(s))\,ds\int_0^s q_n(\mathbf{x}_1(r)-\mathbf{x}_2(r))\,dr\right]\\
&\quad=2\int_0^t ds\int_0^s dr\int \pi_*(x)\,dx\int p[2r, y-x]\,q_n(y)\,dy\\
&\qquad\times\int p[2(s-r), z-y]\,q_n(z)\,dz\\
&\quad=2\int_0^t ds\int q_n(y)\,dy\int q_n(z)\,dz\int p[2r, x-y]\,\pi_*(x)\,dx\\
&\qquad\times\int_0^s dr\, p[2(s-r), z-y]\\
&\quad\leqslant 2\,\|\pi_0\|_\infty\int_0^t ds\int q_n(y)\,dy\int q_n(z)\,dz\, G_s(z-y)
\end{aligned}\tag{3.5}$$

where $G_t(z)=\int_0^t ds\, p[2s, z]$.

Then, to get the assertion, we have to use the same kind of estimate as in Lemma 5.2 and to take into account that $q_n(x) = (a_n)^d q(a_n x)$ and the growth conditions (2.8) on $\{a_n\}$.

The terms $\mathbf{I}_2$ and $\mathbf{I}_3$ converge to zero; indeed, we have

$$\begin{aligned}
&\mathbf{E}[q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \eta_j(s) \mid \mathbf{x}_i] \\
&\quad = \int q_n(z - \mathbf{x}_i(s))\, u(s, z)\, dz \qquad \text{(by Feynman–Kac formula, after integrating over } \sigma_j\text{)} \\
&\quad = \int q(y)\, u(s, y/a_n + \mathbf{x}_i(s))\, dy \leqslant c\, \|\pi_0\|_\infty
\end{aligned} \tag{3.6}$$

Therefore, for any $t \in [0, T]$,

$$\left| \mathbf{E}\left[\int_0^t q_n(\mathbf{x}_j(s) - \mathbf{x}_i(s))\, \eta_j(s)\, ds \,\middle|\, \mathbf{x}_i \right] - c \int_0^t u(s, \mathbf{x}_i(s))\, ds \right|$$
$$\leqslant \int_0^T ds \int \omega(u(s, \cdot), |y|/a_n)\, q(y)\, dy$$

where $\omega(f, \delta) = \sup_{|x-y| \leqslant \delta} |f(x) - f(y)|$ is the modulus of continuity of a function f.

The term $\mathbf{I}_4$ converges to zero trivially by the boundedness of u.

4. PROOF OF THE THEOREM

The proof of propagation of chaos can be reduced to the convergence of the risk functionals:

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}[|R_n(t, \mathbf{x}_i) - R(t, \mathbf{x}_i)|] \to 0 \qquad \text{as} \quad n \to \infty \tag{4.1}$$

namely, we get relation (2.9), observing that, for any $\varepsilon > 0$,

$$\int_0^T \mathbf{E}[|\xi_{i,n}(t) - \xi_i(t)|]\, dt \leqslant 1/\varepsilon \int_0^T \mathbf{E}[|R_n(t, \mathbf{x}_i) - R(t, \mathbf{x}_i)|]\, dt + 2T\varepsilon$$

by Lemma 3.1, setting $S_1(t) = R(t, \mathbf{x}_i)$, $S_2(t) = R_n(t, \mathbf{x}_i)$, and $\sigma = \sigma_i$.

Unfortunately, it is not possible to prove (4.1) directly, so we will have to introduce some kind of "chain or tree of interaction" (see Definition 4.1 below) in a way inspired by Picard's scheme of approximation applied to the nonlinear limit equation (2.7), $(\partial/\partial t) u = 1/2\, \Delta u - c u^2$. Define $u^{(k)}$ as the solution of

$$\begin{aligned}
&(\partial/\partial t) u^{(k)} = 1/2\, \Delta u^{(k)} - c u^{(k-1)} u^{(k)} \qquad k \geqslant 0 \\
&u^{(k)}(0) = \pi_0 \qquad \text{with} \quad u^{(-1)}(t) \equiv 0
\end{aligned} \tag{4.2}$$

then, by standard analysis (see, e.g., ref. 6)

$$\lim_{k \to \infty} \int_0^T \|u^{(k)}(s) - u(s)\|_\infty \, ds = 0 \tag{4.3}$$

where u is the solution of (2.7).

The solution $u^{(k)}(t, x)$ of (4.2) can be interpretated as the probability density to find at time t, in x, a particle, if it dies with rate $cu^{(k-1)}$ or, in other words, if it feels the risk

$$R^{(k)}(t, \mathbf{x}_i) = c \int_0^t u^{(k-1)}(s, \mathbf{x}_i(s)) \, ds \tag{4.4}$$

We note that (4.3) implies that

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}[|R^{(k)}(t, \mathbf{x}_i) - R(t, \mathbf{x}_i)|] \to 0 \qquad \text{as} \quad k \to \infty \tag{4.5}$$

We reformulate this procedure in the n-interacting-particles model.

Definition 4.1. "*Risk at level k*" $R_n^{(k)}(t, \mathbf{x}_j)$:

$$\begin{aligned} &R_n^{(0)}(t, \mathbf{x}_j) \equiv 0, \qquad j = 1,..., n \\ &\xi_{j,n}^{(k)}(t) = \mathbf{I}_{\{R_n^{(k)}(t, \mathbf{x}_j) < \sigma_j\}}, \qquad k \geqslant 0 \\ &R_n^{(k)}(t, \mathbf{x}_j) = 1/n \sum_{h \neq j} \int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_j(s)) \, \xi_{h,n}^{(k-1)}(s) \, ds, \qquad k \geqslant 1 \end{aligned} \tag{4.6}$$

Note that for $k = 1$, $R_n^{(1)}(t, \mathbf{x}_j)$ is the risk felt by the jth particle moving in an environment where all the particles move independently and never die, since $\xi_{j,n}^{(0)}(t) \equiv 1$. So at level 1 we have easily propagation of chaos (apply Lemma 3.3 with $v \equiv 0$).

Among the risks $R_n^{(k)}$ at different levels there is a useful relation, the "sandwiching" property:

Proposition 4.1. For any fixed n, $t \in [0, T]$, $k \geqslant 1$, and $j = 1,..., n$,

$$R_n^{(2k-2)}(t, \mathbf{x}_j) \leqslant R_n^{(2k)}(t, \mathbf{x}_j) \leqslant R_n(t, \mathbf{x}_j) \leqslant R_n^{(2k+1)}(t, \mathbf{x}_j) \leqslant R_n^{(2k-1)}(t, \mathbf{x}_j) \tag{4.7}$$

Proof. Since $\xi_{j,n}^{(0)} = 1$, using (3.2) and (4.6), we get

$$R_n^{(0)}(t, x_j) \leqslant R_n(t, x_j) \leqslant R_n^{(1)}(t, x_j)$$

The result follows by recurrence on k and observing that if S, R, and U are nonnegative functions such that

$$S(t) \leqslant R(t) \leqslant U(t)$$

then

$$\mathbf{I}_{\{S(t) < \sigma\}} \geqslant \mathbf{I}_{\{R(t) < \sigma\}} \geqslant \mathbf{I}_{\{U(t) < \sigma\}}, \qquad \forall \sigma \geqslant 0$$

The introduction of the levels and the "sandwiching" property (4.7) help us to prove that (4.1), namely convergence of R_n to R, is verified: for n fixed, $j = 1,..., n$, and $t \in [0, T]$, we have

$$\begin{aligned} &|R_n(t, \mathbf{x}_j) - R(t, \mathbf{x}_j)| \\ &\quad \leqslant |R_n^{(2k+1)}(t, \mathbf{x}_j) - R^{(2k+1)}(t, \mathbf{x}_j)| \\ &\qquad + |R_n^{(2k)}(t, \mathbf{x}_j) - R^{(2k)}(t, \mathbf{x}_j)| + |R^{(2k+1)}(t, \mathbf{x}_j) - R(t, \mathbf{x}_j)| \\ &\qquad + |R^{(2k)}(t, \mathbf{x}_j) - R(t, \mathbf{x}_j)| \end{aligned} \tag{4.8}$$

First we take k big enough and fix it so that the last two summands are small [see (4.5)]. Therefore the result is achieved by the convergence of $R_n^{(k)}$ to $R^{(k)}$, i.e., once we prove that for $k \geqslant 1$ and for any $j \geqslant 1$

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}[|R_n^{(k)}(t, \mathbf{x}_j) - R^{(k)}(t, \mathbf{x}_j)|] \to 0 \qquad \text{as} \quad n \to \infty \tag{4.9}$$

To summarize the scheme of the proof of the theorem, we draw the following diagram:

$$\begin{tikzcd} R_n^{(2k)} \arrow[d] \arrow[r, phantom, "\leqslant"] & R_n \arrow[d] \arrow[r, phantom, "\leqslant"] & R_n^{(2k+1)} \arrow[d] \\ R^{(2k)} \arrow[r] & R & R^{(2k+1)} \arrow[l] \end{tikzcd}$$

To prove the central arrow, i.e., (4.1), it is sufficient to prove all the other arrows. Note that the horizontal arrows are proven in (4.5).

We need also to introduce the "independent risk" $S_n^{(k)}(t, \mathbf{x}_j)$, which is the risk that the particle $\mathbf{x}_j$ feels in an environment where all the other particles are independent and each one dies with the risk $R^{(k-1)}$ defined in (4.4).

Definition 4.2. *"Independent risk"* $S_n^{(k)}(t, \mathbf{x}_j)$ for any n and $t \in [0, T]$:

$$\begin{aligned} &\eta_j^{(0)}(t) = 1, \qquad j = 1,..., n \\ &\eta_j^{(k)}(t) = \mathbf{I}_{\{R^{(k)}(t, \mathbf{x}_j) < \sigma_j\}}, \qquad k \geqslant 1 \end{aligned} \tag{4.10}$$

$$S_n^{(k)}(t, \mathbf{x}_j) = 1/n \sum_{h \neq j} \int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_j(s))\, \eta_h^{(k-1)}(s)\, ds, \qquad k \geqslant 1$$

In this setting the proof of (4.9) can be divided into two steps thanks to the triangular inequality:

$$\begin{aligned} &|R_n^{(k)}(t, \mathbf{x}_j) - R^{(k)}(t, \mathbf{x}_j)| \\ &\quad \leqslant |S_n^{(k)}(t, \mathbf{x}_j) - R^{(k)}(t, \mathbf{x}_j)| + |R_n^{(k)}(t, \mathbf{x}_j) - S_n^{(k)}(t, \mathbf{x}_j)| \end{aligned} \tag{4.11}$$

The convergence to zero of the first term in the rhs of (4.11) is proven by taking into account that the independent risk $S_n^{(k)}(t, \mathbf{x}_j)$ is the same risk considered in Lemma 3.3 with the function $v = cu^{(k-2)}$ and the definition (4.4) of $R^{(k)}(t, \mathbf{x}_j)$,

$$R^{(k)}(t, \mathbf{x}_j) = c \int_0^t u^{(k-1)}(s, \mathbf{x}_j(s))\, ds$$

where $u^{(k-1)}$ is the solution of

$$(\partial/\partial t)\, u^{(k-1)} = 1/2\, \Delta u^{(k-1)} - cu^{(k-2)} u^{(k-1)}$$

$$u^{(k-1)}(0, x) = \pi_0(x)$$

more precisely,

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}[|S_n^{(k)}(t, \mathbf{x}_j) - R^{(k)}(t, \mathbf{x}_j)|^2] \to 0 \qquad \text{as} \quad n \to \infty \tag{4.12}$$

The convergence of the second term in the rhs of the inequality (4.11) is proven in a slightly stronger form. Before going through the technicalities it is convenient to discuss the difficulties arising in the proof. They come from the fact that even if $R_n^{(k)}(t, \mathbf{x}_j)$ does not depend directly on $\xi_{j,n}^{(k-1)}$ (and therefore on σ_j), it depends indirectly on $\xi_{j,n}^{(k-1)}$ (and therefore on σ_j) through $\xi_{j,n}^{(k-1)}$, $i \neq j$. This is the reason for introducing the "test risk" $Q_n^{(k)}(t, \mathbf{x}_i, \{j\})$ of the ith particle with respect to the particle $\mathbf{x}_j$, i.e., the risk, at the k level, that the particle $\mathbf{x}_i$ feels in an environment where all the particles except $\mathbf{x}_j$ interact among each other but never interact with $\mathbf{x}_j$. We will refer to $\mathbf{x}_j$ as a test particle.

We give here a formal definition for a more general case, in which there is a finite number of test particles.

Definition 4.3. "*Test risk*" $Q_n^{(k)}(t, \mathbf{x}_j, \Gamma)$ with respect to Γ, a finite subset of positive integers. For any n and $t \in [0, T]$

$$\gamma_{j,n}^{(k,\Gamma)}(t) = 0, \qquad k \geqslant 0, \quad j \in \Gamma$$

$$\gamma_{j,n}^{(0)}(t) = 1, \qquad j \in \Gamma^c$$

$$Q_n^{(k)}(t, \mathbf{x}_j, \Gamma) = 1/n \sum_{h \neq j} \int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_j(s))\, \gamma_{h,n}^{(k-1,\Gamma)}(s)\, ds \tag{4.13}$$

$$k \geqslant 1, \qquad \text{for all} \quad j = 1, \ldots, n$$

$$\gamma_{j,n}^{(k,\Gamma)}(t) = \mathbf{I}_{\{Q_n^{(k)}(t, \mathbf{x}_j, \Gamma) < \sigma_j\}}, \qquad k \geqslant 1, \quad j \in \Gamma^c$$

Remark. The test risk with respect to $\{j\}$, $Q_n^{(k)}(t, \mathbf{x}_i, \{j\})$, does not depend on σ_j, neither directly nor indirectly. The same holds in the general case for $Q_n^{(k)}(t, \mathbf{x}_i, \Gamma)$ and σ_j, for $j \in \Gamma$. Moreover, it is obvious that $Q_n^{(k)}(t, \mathbf{x}_i, \varnothing) = R_n^{(k)}(t, \mathbf{x}_i)$.

Therefore, in order to prove (4.9), it is sufficient to prove

$$\sup_{0 \leqslant t \leqslant T} \mathbf{E}[|Q_n^{(k)}(t, \mathbf{x}_i, \Gamma) - S_n^{(k)}(t, \mathbf{x}_i)|] \to 0 \qquad \text{as} \quad n \to \infty \tag{4.14}$$

for any $k \geqslant 1$, for any i, and for any set Γ. The proof is done by induction.

For $k = 1$, just observe that

$$|Q_n^{(1)}(t, \mathbf{x}_i, \Gamma) - S_n^{(1)}(t, \mathbf{x}_i)| = 1/n \sum_{h \neq i, h \in \Gamma} \int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s))\, ds$$

the expectation of which quantity is estimated by $(|\Gamma|\, c\, \|\pi_0\|_\infty\, t)/n$ [see, e.g., (3.6) in Lemma 3.3].

Let us suppose that (4.14) holds for k and any i and any set Γ, then

$$\begin{aligned}
&\mathbf{E}[|Q_n^{(k+1)}(t, \mathbf{x}_j, \Gamma) - S_n^{(k+1)}(t, \mathbf{x}_i)|] \\
&\quad \leqslant 1/n \sum_{h \neq i, h \in \Gamma} \int_0^t \mathbf{E}[q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s))\, \eta_h^{(k)}(s)]\, ds \\
&\qquad + 1/n \sum_{h \neq i, h \in \Gamma^c} \mathbf{E}\left[\int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s))\, |\gamma_{h,n}^{(k,\Gamma)}(s) - \eta_h^{(k)}(s)|\, ds\right]
\end{aligned}$$

The first sum is bounded by $(|\Gamma|\, c\, \|\pi_0\|_\infty\, t)/n$ as in the case $k = 1$.

The second sum, for a fixed $h \neq i$ and $h \in \Gamma^c$, and for any $\varepsilon > 0$, is less than or equal to

$$\begin{aligned}
\mathbf{E}\bigg[\int_0^t & q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s))\{1/\varepsilon\, |Q_n^{(k)}(s, \mathbf{x}_h, \Gamma) - Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})| \\
&+ 1/\varepsilon\, |R^{(k)}(s, \mathbf{x}_h) - Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})| \\
&+ \mathbf{I}_{\{|Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i,h\}) - \sigma_h| \leqslant \varepsilon\}}\}\, ds\bigg]
\end{aligned}$$

Here we have used Lemma 3.2 with $S_1(t) = Q_n^{(k)}(t, \mathbf{x}_h, \Gamma)$, $S_2(t) = R^{(k)}(t, \mathbf{x}_h)$, and $S(t) = Q_n^{(k)}(t, \mathbf{x}_h, \Gamma \cup \{i, h\})$.

Now

$$1/\varepsilon\, \mathbf{E}\left[\int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s))\, |Q_n^{(k)}(s, \mathbf{x}_h, \Gamma) - Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})|\, ds\right]$$

converges to zero, by Lemma 5.3, with $\Lambda = \{i, h\}$.

Now observe that $R^{(k)}(s, \mathbf{x}_h)$ and $Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})$ depend only on $\mathbf{x}_h$ and on $(\mathbf{x}_j, \sigma_j)$ for $j \neq i, h$ [here is the reason why we need to use $Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})$]. Hence

$$\mathbf{E}\left[\int_0^t q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s))\{1/\varepsilon \, |R^{(k)}(s, \mathbf{x}_h) - Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})| \right.$$

$$\left. + \mathbf{I}_{\{|Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i,h\}) - \sigma_h| \leqslant \varepsilon\}}\} \, ds\right]$$

$$= \mathbf{E}\left[\int_0^t \mathbf{E}[q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s)) \,|\, \mathbf{x}_h; (\mathbf{x}_j, \sigma_j) \text{ for } j \neq i, h] \right.$$

$$\times \{1/\varepsilon \, |R^{(k)}(s, \mathbf{x}_h) - Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})|$$

$$\left. + \mathbf{I}_{\{|Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i,h\}) - \sigma_h| \leqslant \varepsilon\}}\} \, ds\right]$$

which, using the fact that

$$\mathbf{E}[q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s)) \,|\, \mathbf{x}_h; (\mathbf{x}_j, \sigma_j) \text{ for } j \neq i, h]$$
$$= \mathbf{E}[q_n(\mathbf{x}_h(s) - \mathbf{x}_i(s)) \,|\, \mathbf{x}_h] \leqslant c \, \|\pi_0\|_\infty$$

[see (3.6)] and the independence of $Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})$ and σ_h, is less than or equal to

$$c \, \|\pi_0\|_\infty \int_0^t \{1/\varepsilon \, \mathbf{E}[|R^{(k)}(s, \mathbf{x}_h) - S_n^{(k)}(s, \mathbf{x}_h)|]$$

$$+ 1/\varepsilon \, \mathbf{E}[|S_n^{(k)}(s, \mathbf{x}_h) - Q_n^{(k)}(s, \mathbf{x}_h, \Gamma \cup \{i, h\})|] + 2\varepsilon\} \, ds$$

which converges to zero uniformly in $t \in [0, T]$, by (4.12), the induction hypothesis, and the arbitrariness of ε.

5. TECHNICAL LEMMAS

The following lemma is a generalization of relation (3.5) of Lemma 3.3.

Lemma 5.1. Let $\mathbf{x}_i$, $i \geqslant 1$, be independent Brownian motions in $\mathbf{R}^d$, each with initial probability density $\pi_0 \in \mathbf{L}^\infty$, and let g be a nonnegative function belonging to $\mathbf{L}^1$. Set

$$\mathbf{J}_m(h, t) = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \, g(\mathbf{x}_1(s_1) - \mathbf{x}_2(s_1))$$

$$\times \, g(\mathbf{x}_2(s_2) - \mathbf{x}_3(s_2)) \cdots g(\mathbf{x}_m(s_m) - \mathbf{x}_h(s_m)) \qquad (5.1)$$

Then, if $h > m$,

$$\mathbf{E}[\mathbf{J}_m(h, t)] \leqslant (\|\pi_0\|_\infty \|g\|_1 t)^m/(m)! \tag{5.2}$$

and if $1 \leqslant h < m$, $m \geqslant 2$,

$$\mathbf{E}[\mathbf{J}_m(h, t)] \leqslant (\|\pi_0\|_\infty t)^{m-1}/(m-1)! (\|g\|_1)^{m-2} \times \sup_{x \in \mathbf{R}^d} \int g(y)\, dy \int g(z)\, dz\, G_t(z+y+x) \tag{5.3}$$

where $G_t(z) = \int_0^t ds\, p[2s, z]$ and $p[s, y-x]$ is the transition probability of a standard Brownian motion.

Proof. First we note that, if $\mathfrak{A}$ is a σ-algebra independent of $\mathbf{x}_i$, $i \leqslant m$, and such that $\mathbf{x}_h$ is $\mathfrak{A}$-measurable, with $h > m$, then

$$\mathbf{E}[g(\mathbf{x}_1(s_1) - \mathbf{x}_2(s_1))\, g(\mathbf{x}_2(s_2) - \mathbf{x}_3(s_2)) \cdots g(\mathbf{x}_m(s_m) - \mathbf{x}_h(s_m)) \mid \mathfrak{A}] \leqslant (\|\pi_0\|_\infty \|g\|_1)^m \tag{5.4}$$

because of the independence of $\mathbf{x}_i - \mathbf{x}_{i+1}$ and $\mathfrak{A}$, and since [see (3.6)]

$$\mathbf{E}[g(\mathbf{x}_m(s_m) - \mathbf{x}_h(s_m)) \mid \mathfrak{A}] = \mathbf{E}[g(\mathbf{x}_m(s_m) - \mathbf{x}_h(s_m)) \mid \mathbf{x}_h] \leqslant \|\pi_0\|_\infty \|g\|_1$$

This observation implies immediately (5.2) and allows us to reduce the proof of (5.3) to the case $h = 1$. Indeed, taking the conditional expectation with respect to $\mathfrak{A} = \sigma(\mathbf{x}_h, \mathbf{x}_{h+1}, \dots, \mathbf{x}_m)$ and using (5.4) with $m = h-1$, we get that

$$\mathbf{E}[\mathbf{J}_m(h, t)] \leqslant \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{h-2}} ds_{h-1} \times (\|\pi_0\|_\infty \|g\|_1)^{h-1}\, \mathbf{E}[\mathbf{J}_{m-h+1}(1, s_{h-1})]$$

When $h = 1$ we will prove that

$$\mathbf{E}[\mathbf{J}_m(1, t)] \leqslant (\|\pi_0\|_\infty t)^{m-1}/(m-1)! \times \int g(z_1)\, dz_1 \int g(z_2)\, dz_2 \cdots \int g(z_m)\, dz_m \times G_t(z_1 + z_2 + \cdots + z_m) \tag{5.5}$$

which immediately implies (5.3).

We can write, setting $\pi(s, x)$ the probability density of the random variable $\mathbf{x}_i(s)$ for any $i \geqslant 1$,

$$\begin{aligned}
&\mathbf{E}[g(\mathbf{x}_1(s_1)-\mathbf{x}_2(s_1))\, g(\mathbf{x}_2(s_2)-\mathbf{x}_3(s_2)) \cdots g(\mathbf{x}_m(s_m)-\mathbf{x}_1(s_m))] \\
&\quad = \int dx_1 \int dy_1 \int dx_2 \int dy_2 \cdots \int dx_m \int dy_m \\
&\qquad \times \pi(s_m, x_1)\, p[s_1 - s_m, y_1 - x_1]\, g(y_1 - y_2) \\
&\qquad \times \left\{ \bigotimes_{i=2,m-1} \pi(s_i, x_i)\, p[s_{i-1} - s_i, y_i - x_i]\, g(x_i - y_{i+1}) \right\} \\
&\qquad \times \pi(s_m, x_m)\, p[s_{m-1} - s_m, y_m - x_m]\, g(x_m - x_1)
\end{aligned}$$

setting $y_1 - y_2 = z_1$, $x_i - y_{i+1} = z_i$ for $2 \leqslant i \leqslant m-1$, $x_m - x_1 = z_m$, and taking into account that $\pi(s, x)$ is bounded by $\|\pi_0\|_\infty$, we get that the expression above is less than or equal to

$$\begin{aligned}
&(\|\pi_0\|_\infty)^{m-1} \int dx_1 \int dz_1 \int dz_2 \int dy_2 \cdots \int dz_m \int dy_m\, \pi(s_m, x_1) \\
&\quad \times \bigotimes_{i=1,m} g(z_i)\, p[s_1 - s_m, z_1 + y_2 - x_1] \\
&\quad \times \bigotimes_{i=2,m-1} p[s_{i-1} - s_i, y_i - z_i - y_{i+1}]\, p[s_{m-1} - s_m, y_m - z_m - x_1]
\end{aligned}$$

Now integrating with respect to $dy_3 \cdots dy_m$ and using Chapman–Kolmogorov equality, we get that this expression is equal to

$$\begin{aligned}
&(\|\pi_0\|_\infty)^{m-1} \int dz_1 \int dz_2 \cdots \int dz_m \bigotimes_{i=1,m} g(z_i) \int dx_1\, \pi(s_m, x_1) \\
&\qquad \times \int dy_2\, p[s_1 - s_m, z_1 + y_2 - x_1]\, p\left[s_1 - s_m, y_2 - \sum_{i=2,m} z_i - x_1\right] \\
&\quad = (\|\pi_0\|_\infty)^{m-1} \int dz_1 \int dz_2 \cdots \int dz_m \bigotimes_{i=1,m} g(z_i) \int dx_1 \\
&\qquad \times \pi(s_m, x_1)\, p\left[2(s_1 - s_m), \sum_{i=1,m} z_i\right]
\end{aligned}$$

which, integrating in ds_m over $[0, s_1]$, noting that $G_s(z) \leqslant G_t(z)$ for $s \leqslant t$, and then integrating over $ds_1 \cdots ds_{m-1}$ gives (5.3).

Lemma 5.2. Let $g_a(x) = a^d g(ax)$, with $g \in L^1 \cap H^{-1}$ a nonnegative function; then

$$\sup_{x \in \mathbf{R}^d} \int g_a(y)\, dy \int g_a(z)\, G_t(z+y+x)\, dz$$

$$\begin{aligned} &\text{is bounded in } a && \text{if} \quad d=1 \\ &= O(\log a) && \text{if} \quad d=2 \\ &= O(a^{d-2}) && \text{if} \quad d \geqslant 3 \end{aligned}$$

Proof. Let us denote by

$$\hat{f}(\omega) = \int f(x) \exp\{-i\omega x\}\, dx$$

the Fourier transform of f so that

$$\hat{G}_t(\omega) = (1 - \exp\{-t\omega^2\})/\omega^2$$

Set $h_a(x) = g_a(-x)$, so that $\hat{h}_a(\omega) = \hat{g}_a(-\omega)$. We can rewrite

$$\begin{aligned} &\int g_a(y)\, dy \int g_a(z)\, dz\, G_t(z+y+x) \\ &\quad = (h_a * (h_a * G_t))(x) \\ &\quad = \int (\hat{g}_a)^2 (-\omega) \exp\{i\omega x\}(1 - \exp\{-t\omega^2\})/\omega^2\, d\omega \\ &\quad \leqslant \int |\hat{g}_a(\omega)|^2 (1 - \exp\{-t\omega^2\})/\omega^2\, d\omega \end{aligned} \tag{5.6}$$

Observe that $|\hat{g}_a(\omega)| = |\hat{g}(\omega/a)|$ is bounded by $\|g\|_1$.

So if $d=1$, the quantity $\int g_a(y)\, dy \int g_a(z)\, dz\, G_t(z+y+x)$ is bounded uniformly in a, since the function $\omega \to (1 - \exp\{-t\omega^2\})/\omega^2$ is integrable in $\mathbf{R}^1$:

$$\begin{aligned} &\int |\hat{g}_a(\omega)|^2 (1 - \exp\{-t\omega^2\})/\omega^2\, d\omega \\ &\quad \leqslant (\|g\|_1)^2 \left\{ \int_{|\omega| \leqslant 1} (1 - \exp\{-t\omega^2\})/\omega^2\, d\omega \right. \\ &\qquad \left. + \int_{|\omega| > 1} (1 - \exp\{-t\omega^2\})/\omega^2\, d\omega \right\} \\ &\quad \leqslant (\|g\|_1)^2 \left(2t + \int_{|\omega| > 1} 1/\omega^2\, d\omega \right) \end{aligned}$$

For $d \geqslant 2$ we change variable $\bar{\omega} = \omega/a$ and rewrite the last member of the inequality (5.6) as

$$a^{d-2} \int |\hat{g}_a(\bar{\omega})|^2 \, [1 - \exp\{-t(a\bar{\omega})^2\}]/\bar{\omega}^2 \, d\bar{\omega}$$

If $d = 2$, we divide the integral into three parts according to $|\bar{\omega}| \leqslant 1/a$, $1/a < |\bar{\omega}| \leqslant 1$, and $|\bar{\omega}| > 1$:

$$\begin{aligned}
&\int_{|\bar{\omega}| \leqslant 1/a} |\hat{g}_a(\bar{\omega})|^2 \, [1 - \exp\{-t(a\bar{\omega})^2\}]/\bar{\omega}^2 \, d\bar{\omega} \\
&\quad + \int_{1/a < |\bar{\omega}| \leqslant 1} |\hat{g}_a(\bar{\omega})|^2 \, [1 - \exp\{-t(a\bar{\omega})^2\}]/\bar{\omega}^2 \, d\bar{\omega} \\
&\quad + \int_{|\bar{\omega}| > 1} |\hat{g}_a(\bar{\omega})|^2 \, [1 - \exp\{-t(a\bar{\omega})^2\}]/\bar{\omega}^2 \, d\bar{\omega} \\
&\leqslant \pi \, 1/a^2 \, \{ta^2(\|g\|_1)^2\} + (\|g\|_1)^2 \int_{1/a < |\bar{\omega}| \leqslant 1} 1/\bar{\omega}^2 \, d\bar{\omega} \\
&\quad + \int_{|\bar{\omega}| > 1} |\hat{g}_a(\bar{\omega})|^2 \, 1/\bar{\omega}^2 \, d\bar{\omega} \\
&\leqslant \pi t(\|g\|_1)^2 + b_2(\|g\|_1)^2 \log(a) + 2 \int_{|\bar{\omega}| > 1} |\hat{g}_a(\bar{\omega})|^2 \, (\bar{\omega}^2 + 1)^{-1} \, d\bar{\omega} \\
&\leqslant \pi t(\|g\|_1)^2 + b_2(\|g\|_1)^2 \log(a) + 2(\|g\|_{H^{-1}})^2
\end{aligned}$$

If $d \geqslant 3$, we divide the integral into two parts according to $|\bar{\omega}| \leqslant 1$ and $|\bar{\omega}| > 1$:

$$\begin{aligned}
&\int_{|\bar{\omega}| \leqslant 1} |\hat{g}_a(\bar{\omega})|^2 \, [1 - \exp\{-t(a\bar{\omega})^2\}]/\bar{\omega}^2 \, d\bar{\omega} \\
&\quad + \int_{|\bar{\omega}| > 1} |\hat{g}_a(\bar{\omega})|^2 \, [1 - \exp\{-t(a\bar{\omega})^2\}]/\bar{\omega}^2 \, d\bar{\omega} \\
&\leqslant (\|g\|_1)^2 \int_{|\bar{\omega}| \leqslant 1} 1/\bar{\omega}^2 \, d\bar{\omega} + \int_{|\bar{\omega}| > 1} |\hat{g}_a(\bar{\omega})|^2 \, 1/\bar{\omega}^2 \, d\bar{\omega} \\
&\leqslant b_d(\|g\|_1)^2 + 2 \int_{|\bar{\omega}| > 1} |\hat{g}_a(\bar{\omega})|^2 \, (\bar{\omega}^2 + 1)^{-1} \, d\bar{\omega} \\
&\leqslant b_d(\|g\|_1)^2 \, \|g\|_1 + 2(\|g\|_{H^{-1}})^2
\end{aligned}$$

where b_d is a universal constant depending on d.

Lemma 5.3. Suppose that $q_n(x)=(a_n)^d q(a_n x)$ [see (2.3)], where $q\in L^1\cap H^{-1}$ is a nonnegative function and a_n is a divergent sequence. If, moreover, when $d\geqslant 2$, $\{a_n\}$ satisfies the growth conditions (2.8):

$$\text{if} \quad d=2, \qquad \log a_n = o(n)$$

$$\text{if} \quad d\geqslant 3, \qquad (a_n)^{d-2}=o(n)$$

Then, for every $k\geqslant 1$ and $m\geqslant 1$, the following relation holds:

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\Bigg[\int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \\ \times q_n(\mathbf{x}_1(s_1)-\mathbf{x}_2(s_1))\, q_n(\mathbf{x}_2(s_2)-\mathbf{x}_3(s_2)) \cdots q_n(\mathbf{x}_m(s_m)-\mathbf{x}_{m+1}(s_m)) \\ \times |Q_n^{(k)}(s_{m+1}, \mathbf{x}_{m+1}, \Gamma)-Q_n^{(k)}(s_{m+1}, \mathbf{x}_{m+1}, \Gamma\cup\Lambda)|\Bigg]=0\end{aligned}$$

whenever $\{1,..., m+1\}$ is contained in $\Gamma\cup\Lambda$ [see (4.11) for the definition of $Q_n^{(k)}(s, \mathbf{x}, \Gamma)$].

Proof. In the sequel we will drop the dependence of the integrands on time to simplify the notations.

For $k=1$ and for every $m\geqslant 1$,

$$\begin{aligned}\mathbf{E}\Bigg[\int_0^t ds_1 \cdots \int_0^{s_{m-1}} ds_m\, q_n(\mathbf{x}_1-\mathbf{x}_2)\cdots q_n(\mathbf{x}_m-\mathbf{x}_{m+1}) \\ \times |Q_n^{(1)}(\mathbf{x}_{m+1}, \Gamma)-Q_n^{(1)}(\mathbf{x}_{m+1}, \Gamma\cup\Lambda)|\Bigg] \\ \leqslant 1/n \sum_{h\in\Lambda} \mathbf{E}\Bigg[\int_0^t ds_1 \cdots \int_0^{s_{m-1}} ds_m\, q_n(\mathbf{x}_1-\mathbf{x}_2)\cdots q_n(\mathbf{x}_m-\mathbf{x}_{m+1}) \\ \times \int_0^{s_m} ds_{m+1}\, q_n(\mathbf{x}_m-\mathbf{x}_h)\Bigg]\end{aligned}$$

Then, if $h\in\Lambda\backslash\{1,..., m+1\}$, we apply (5.2) of Lemma 5.1, while if $h\in\Lambda\cap\{1,..., m+1\}$, we apply Lemma 5.2 and the growth conditions (2.8) on $\{a_n\}$.

Now suppose that the assertion is true for k and any $m\geqslant 1$; then

$$\begin{aligned}\mathbf{E}\Bigg[\int_0^t ds_1 \cdots \int_0^{s_{m-1}} ds_m\, q_n(\mathbf{x}_1-\mathbf{x}_2)\cdots q_n(\mathbf{x}_m-\mathbf{x}_{m+1}) \\ \times |Q_n^{(k+1)}(\mathbf{x}_{m+1}, \Gamma)-Q_n^{(k+1)}(\mathbf{x}_{m+1}, \Gamma\cup\Lambda)|\Bigg]\end{aligned}$$

$$\leqslant 1/n \sum_{h \in A} \mathbf{E}\left[\int_0^t ds_1 \cdots \int_0^{s_{m-1}} ds_m\, q_n(\mathbf{x}_1 - \mathbf{x}_2) \cdots q_n(\mathbf{x}_m - \mathbf{x}_{m+1})\right.$$
$$\left.\times \int_0^{s_m} ds_{m+1}\, q_n(\mathbf{x}_{m+1} - \mathbf{x}_h)\right]$$
$$+ 1/n \sum_{h \in (\Gamma \cup A)^c} \mathbf{E}\left[\int_0^t ds_1 \cdots \int_0^{s_{m-1}} ds_m\, q_n(\mathbf{x}_1 - \mathbf{x}_2) \cdots q_n(\mathbf{x}_m - \mathbf{x}_{m+1})\right.$$
$$\left.\times \int_0^{s_m} ds_{m+1}\, q_n(\mathbf{x}_{m+1} - \mathbf{x}_h)\, |\gamma_{h,n}^{(k,\Gamma)} - \gamma_{h,n}^{(k,\Gamma \cup A)}|\right]$$

The first term in the rhs of this inequality is exactly equal to the case $k = 1$.

By Lemma 3.2, the second term is less or than equal to

$$\mathbf{E}\left[\int_0^t ds_1 \cdots \int_0^{s_{m-1}} ds_m\, q_n(\mathbf{x}_1 - \mathbf{x}_2) \cdots q_n(\mathbf{x}_m - \mathbf{x}_{m+1})\right.$$
$$\times \int_0^{s_m} ds_{m+1}\, q_n(\mathbf{x}_{m+1} - \mathbf{x}_h)$$
$$\times \{1/\varepsilon\, |Q_n^{(k)}(\mathbf{x}_h, \Gamma) - Q_n^{(k)}(\mathbf{x}_h, \Gamma \cup A \cup \{h\})|$$
$$+ 1/\varepsilon\, |Q_n^{(k)}(\mathbf{x}_h, \Gamma \cup A) - Q_n^{(k)}(\mathbf{x}_h, \Gamma \cup A \cup \{h\})|$$
$$\left. + \mathbf{I}_{\{|Q_n^{(k)}(\mathbf{x}_h, \Gamma \cup A \cup \{h\}) - \sigma_h| \leqslant \varepsilon\}}\}\right]$$

The first two addends go to zero using the induction hypothesis for k and for $m+1$ instead of m, noting that $\{1,..., m+1, h\}$ is contained in $\Gamma \cup A \cup \{h\}$ and setting $\mathbf{x}_{m+2} = \mathbf{x}_h$. Finally, the third addend is less than or equal to $\varepsilon(\|\pi_0\|_\infty ct)^{m+1}/(m+1)!$ for any $\varepsilon > 0$. To prove it, we take conditional expectation with respect to $\mathfrak{A} = \sigma\{(\mathbf{x}_j, \sigma_j), j \in (\Gamma \cup A)^c\}$. Then we use (5.4) with $m+1$ instead of m [note that h belongs to $(\Gamma \cup A)^c$ and therefore h does not belong to $\{1,..., m+1\}$, i.e., $h > m+1$] and the fact that $Q_n^{(k)}(\mathbf{x}_h, \Gamma \cup A \cup \{h\})$ and σ_h are independent and measurable with respect to $\mathfrak{A}$.

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